Un-graviton corrections to the Schwarzschild black hole

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Abstract

We introduce an effective action smoothly extending the standard Einstein-Hilbert action to include *un-gravity* effects. The improved field equations are solved for the *Un-graviton corrected* Schwarzschild geometry reproducing the Mureika result. This is an important test to confirm the original "guess" of the form of the Un-Schwarzschild metric. Instead of working in the weak field approximation and "dressing" the Newtonian potential with un-gravitons, we solve the "effective Einstein equations" including all order un-gravity effects. An unexpected "bonus" of accounting un-gravity effects is the *fractalisation* of the event horizon. In the un-gravity dominated regime the event horizon thermodynamically behaves as fractal surface of dimensionality twice the scale dimension d_U.

1 Introduction

Scale invariance plays an important role in different sectors of modern theoretical physics. In statistical mechanics fluctuations occur at all scale near a

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critical point and phase transitions require a scale invariant description.
In field theory scale invariance is related to the behavior of the theory under dilatations and indicates the absence of a fundamental length scale.
For later convenience, it can be useful to recall that in Mathematics scale invariance is tightly linked to the self-similarity of fractal curves and surfaces.
Finally, scale invariant quantum field theories generally describes massless particles. Only, recently a new implementation of this symmetry has been suggested [1,2] as a consequence of non-trivial fixed point in the infrared regime [3]. In this new framework the intuitive notion of scale invariance as a property of massless particles only, has been extended to a new kind of stuff with no definite mass at all. For this reason, this presently unknown, scale invariant, sector of the elementary particle spectrum has been dubbed as the *un-particle* sector.
The proposal is that below some critical energy scale Λ_U the standard model particles can interact with un-particles. The forthcoming start of LHC activity has focused the interest of the high energy physics community on possible experimental signature of un-particles events at the TeV energy scale [4,5,6,7,8] [9,10,11,12] [24,25]. Astroparticle and cosmological un-particle effects have been considered as well [13,14,15,16], [17,18,19,20,21], [22,23].
On the theoretical side, interesting connections between un-particles and non-standard Kaluza-Klein dynamics in extra-dimensions, and AdS/CFT duality, have been pointed out in [26,27,28,29,30]. Even deeper connections between un-gravity and trans-Planckian physics are currently under investigation [31,32]. Finally, two of us (P.G. and E.S.) have built an "effective action" for several kind of un-particle fields, including un-gravitons [33] . Encoding un-gravity into an effective action can be useful in view of studying gravitational effects [34,35] beyond the weak field approximation.
In this letter we show that the un-Schwarzschild metric guessed in [35] through perturbative arguments is an *exact* solution of the field equations obtained from the effective theory introduced in [33]. A non-trivial step in obtaining the solution is the introduction of a point-like source in the un-gravity Einstein equations.
In the final part of this communication we recover "from scratch" the area law for the un-Schwarzschild black hole. The resulting expression for the area suggests the horizon to be a *fractal* surface of dimension twice the scale dimension d_U.

2 Un-gravity field equations

The physical system we are going to investigate is an "hybrid" of classical matter, classical gravity, and "quantum" un-gravity due to the exchange of un-gravitons. The action for this system is the sum of a classical functional

S_M for matter, and a *non-local* effective action S_U smoothly extending the Einstein-Hilbert action to include un-gravitons dynamics.

$$S \equiv S_M + S_U \tag{1}$$

S_M is the classical matter action for a massive, point-like, particle "sitting" in the origin. There is some freedom to choose the explicit for of this functional. Simplicity suggests to introduce S_M in the form of the action for pressure-less, static fluid, with a "singular" (but integrable !) energy density mimicking a "point-mass":

$$S_M \equiv -\int d^4x\sqrt{g}\,\rho\,(\,x\,)\;u^\mu\,u^\nu\;,\quad \rho\,(\,x\,) \equiv \frac{M}{\sqrt{g}}\int d\tau\,\delta\,(\,x - x\,(\tau)\,) \tag{2}$$

The un-gravity action is obtained by combining the Einstein-Hilbert functional and the non-local effective action we obtained in [33] :

$$S_U = \frac{1}{2\kappa^2}\int d^4x\sqrt{g}\left[1 + \frac{A_{d_U}}{(\,2d_U - 1\,)\sin\,(\,\pi\,d_U\,)}\frac{\kappa_*^2}{\kappa^2}\left(\frac{-D^2}{\Lambda_U^2}\right)^{1-d_U}\right]^{-1} R \tag{3}$$

where, D^2 is the generally covariant D'Alembertian;

$$A_{d_U} \equiv \frac{16\pi^{5/2}}{(\,2\pi\,)^{2d_U}}\frac{\Gamma\,(\,d_U + 1/2\,)}{\Gamma\,(\,d_U - 1\,)\,\Gamma\,(\,2d_U\,)} \tag{4}$$

while κ_* represents the un-gravitational Newton constant

$$\kappa_* \equiv \frac{1}{\Lambda_U}\left(\frac{\Lambda_U}{M_U}\right)^{d_{UV}} \tag{5}$$

$$\simeq \frac{1}{\Lambda_U}\left(\frac{\Lambda_U}{M_U}\right)\;,\qquad d_{UV} \simeq 1 \tag{6}$$

The strength of coupling constant is determined by the mass scale M_U which replaces the Planck mass. $\kappa = 16\pi\,G_N = M_{Pl.}^{-2}$.

Our main purpose is to solve the field equations derived from S by assuming the source is static, i.e. the four-velocity field u^μ has only non-vanishing time-like component

$$u^\mu \equiv \left(u^0\;, \vec{0}\right)\;,\quad u^0 = \frac{1}{\sqrt{-g^{00}}} \tag{7}$$

Einstein equations are obtained by varying the action (3) with respect to the metric $g_{\mu\nu}$. By neglecting surface terms coming from the variation of the generally covariant D'Alembertian, we find

$$R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\, R = \kappa^2\left[1+\frac{A_{d_U}\Lambda_U^{2-2d_U}}{(\,2d_U-1\,)\,\sin(\,\pi\, d_U\,)}\frac{\kappa_*^2}{\kappa^2}\,(\,-D\,)^{d_U-1}\right]T^\mu{}_\nu$$
$$\equiv \kappa^2\, T^\mu{}_\nu + \kappa_*^2\frac{A_{d_U}}{\sin(\,\pi\, d_U\,)}T_U{}^\mu_\nu \tag{8}$$

In Eq.(8) we have "shifted" the un-particle terms to the r.h.s. leaving the l.h.s. in the canonical form. As a matter of fact, Eq. (8) can be seen as 'ordinary" gravity coupled to an "exotic" source term, instead of un-gravity produced by an ordinary particle. The two interpretations are physically equivalent.
Now, the reader is probably thinking "... but, in the Einstein field equations for the Schwarzschild solution there is no energy-momentum tensor. Schwarzschild geometry is a *vacuum* solution." It is a quite common misunderstanding to think that vacuum means "in the absence of a source" instead of *outside* a compact (localized) source. In the Schwarzschild case, the source is a point-like particle sitting in the origin [4]. The corresponding energy-momentum tensor is given by [41]

$$T^0_0 = -\frac{M}{4\pi\, r^2}\delta(\,r\,) \tag{9}$$
$$T^r_r = 0 \tag{10}$$
$$T^\theta_\theta = T^\phi_\phi = -\frac{M}{16\pi\, r}\delta(\,r\,)\,\frac{1}{g_{00}}\partial_r\, g_{00} \tag{11}$$

where, T^θ_θ , T^ϕ_ϕ are determined by the requirement $\Delta_\mu T^{\mu\nu} = 0$.
With this kind of energy-momentum tensor the 00 and rr components of the

[4] In General Relativity textbooks it is customary to introduce the Schwarzschild solution without even mentioning the presence of a point-like source. Once the Einstein equations are solved in the vacuum, the integration constant is determined by matching the solution with the Newtonian field outside a spherically symmetric mass distribution. Definitely, this is not the most straightforward way to expose students, and not only them, to one of the most fundamental solutions of the Einstein equations. Moreover, the presence of a curvature singularity in the origin, where from the very beginning a finite mass-energy is squeezed into a zero-volume point, is introduced as a "shocking", un-expected result. Against this background, we showed in [36,37,38,39,40] that once quantum delocalization of the source is accounted, all these "flaws" disappear.

metric tensor turn out to be of the form

$$g_{rr}^{-1} = 1 - \frac{2G_N}{r} M(r) = -\frac{e^{-h_0}}{g_{00}} \tag{12}$$

where the constant h_0 can be freely re-absorbed into the definition of the time coordinate, and

$$M(r) \equiv 4\pi \int dr\, r^2\, T_0^0 \tag{13}$$

In Equation (13) the symbol $\int dr$ indicates an indefinite integration. The constant factor e^{h_0} can be safely rescaled to 1 by a redefinition of the time coordinate.

As a first step towards solving the Einstein field equation we need to transform the energy density (9) into its un-particle counterpart $T^0_{U\,0}$. We momentarily switch to isotropic, (free-falling) Cartesian-like coordinate for computational convenience. Finally, it will be easy to transform the result in a spherical frame. By taking into account (8) and(9) we can write

$$T^0_{U\,0} \equiv \rho_U = \frac{M}{2d_U - 1}\, \Lambda_U^{2-2d_U} \left(-\nabla^2\right)^{d_U - 1} \delta(\vec{x}) \tag{14}$$

$$= \frac{M}{2d_U - 1} \frac{\Lambda_U^{2-2d_U}}{(2\pi)^3} \int d^3k \left(\frac{1}{\vec{k}^2}\right)^{1-d_U} e^{i\vec{k}\cdot\vec{x}} \tag{15}$$

In order to proceed we use the Schwinger representation for $\left(1/\vec{k}^{\,2}\right)^{1-d_U}$:

$$\left(\frac{1}{\vec{k}^{\,2}}\right)^{1-d_U} = \frac{1}{\Gamma(1-d_U)} \int_0^\infty ds s^{-d_U}\, e^{-s\,\vec{k}^{\,2}} \tag{16}$$

By inserting (16) in (15), integrating first over $\vec{k}$ and secondly over the Schwinger parameter, we get ρ_U

$$\rho_U(\vec{x}) = \frac{2^{2d_U}}{16\pi^{3/2}} \frac{\Gamma(d_U + 1/2)}{\Gamma(1-d_U)} \frac{M}{2d_U - 1} \Lambda_U^{2-2d_U} \left(\frac{1}{\vec{x}^{\,2}}\right)^{d_U + 1/2} \tag{17}$$

which is manifestly spherically symmetric and can be conveniently written in terms of the radial distance from the origin $r \equiv |\vec{x}|$:

$$\rho_U\,(\,r\,) = \frac{2^{2d_U-1}}{16\pi^{3/2}}\frac{\Gamma\,(\,d_U-1/2\,)}{\Gamma\,(\,1-d_U\,)}\,M\,\Lambda_U^{2-2d_U}\left(\frac{1}{r}\right)^{2d_U+1} \tag{18}$$

Thus,

$$\begin{aligned} M\,(\,r\,) &= 4\pi\,\frac{2^{2d_U-1}}{16\pi^{3/2}}\frac{\Gamma\,(\,d_U-1/2\,)}{\Gamma\,(\,1-d_U\,)}\,M\Lambda_U^{2-2d_U}\int dr\,r^{1-2d_U} \\ &= \frac{2^{2d_U-1}}{4\pi^{1/2}}\frac{\Gamma\,(\,d_U-1/2\,)}{\Gamma\,(\,1-d_U\,)}\,M\Lambda_U^{2-2d_U}\,\frac{r^{2-2d_U}}{2\,(\,1-d_U\,)} \end{aligned} \tag{19}$$

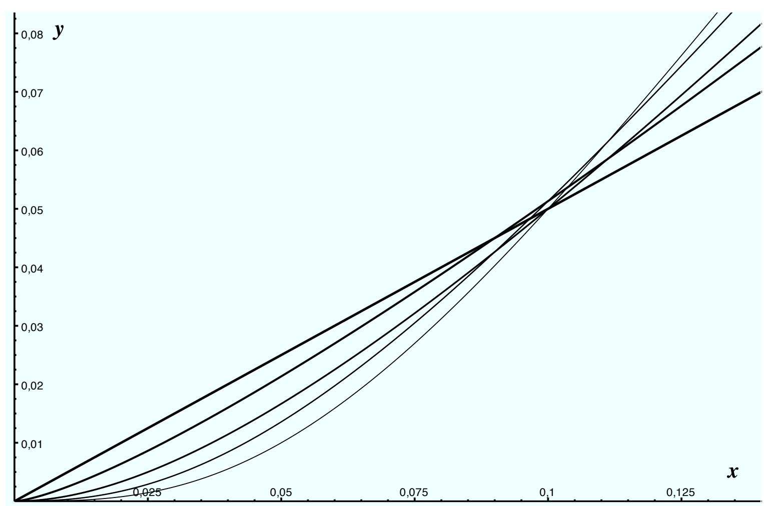

Fig. 1. *Plot of the function $M(r_H)$ for different values of d_U. The straight line corresponds to $d_U = 1$, the other lines have $d_U = 1.25\ ,1.5\ ,1.75\ ,2$.*

Finally,

$$M\,(\,r\,) = \frac{2^{2d_U-2}}{4\pi^{1/2}}\frac{\Gamma\,(\,d_U-1/2\,)}{\Gamma\,(\,2-d_U\,)}\,M\Lambda_U^{2-2d_U}\left(\frac{1}{r}\right)^{2d_U-2} \tag{20}$$

An arbitrary integration constant has been set to zero, as we are interested to study the field determined in a unique way by an un-particle source, and nothing else.
Thus, we find

$$g_{rr}^{-1} = -g_{00} = 1 - \frac{2MG_N}{r}\left[\,1+\kappa_*^2\,\frac{A_{d_U}}{\sin\,(\,\pi\,d_U\,)}\frac{M\,(\,r\,)}{2M\,G_N}\,\right]$$

$$= 1 - \frac{R_s}{r}\left[1 + \frac{M^2_{Pl.}\kappa^2_*}{\pi^{2d_U-1}}\,\Lambda_U^{2-2d_U}\,\frac{\Gamma\,(\,d_U-1/2\,)\,\Gamma\,(\,d_U+1/2\,)}{\Gamma\,(\,2d_U\,)}\,\left(\frac{1}{r}\right)^{2d_U-2}\right]$$

$$= 1 - V_N\,(\,r\,)\left[1 + \Gamma_U\,\left(\frac{R_*}{r}\right)^{2d_U-2}\right] \tag{21}$$

$$\Gamma_U \equiv \frac{2}{\pi^{2d_U-1}}\frac{\Gamma\,(\,d_U-1/2\,)\,\Gamma\,(\,d_U+1/2\,)}{\Gamma\,(\,2d_U\,)} \tag{22}$$

where, $R_s = 2MG_N = 2M/M^2_{Pl.}$ is the Schwarzschild radius; $-V_N\,(\,r\,)$ is the Newton gravitational potential, and the new gravitational length scale R_* is defined as

$$R_* \equiv \frac{1}{\Lambda_U}\left(\frac{M_{Pl.}}{M_U}\right)^{1/(d_U-1)} \tag{23}$$

The metric (21) has been guessed in the weak-field case [35] from the form of the un-graviton dressed Newtonian potential. At the linearized level the correction to the Newton law is of the form

$$V_U\,(\,r\,) = -G_U\frac{m_1 m_2}{r^{2d_U-1}} \tag{24}$$

In ref.[23] the phenomenological consequences of $V_U(r)$ have been discussed and the correction to the perihelion precession of Mercury has been obtained

$$\delta\theta \simeq 2\pi\,\,(d_U-1\,)\,(2d_U-1\,)\,\frac{V_U}{V_N} \tag{25}$$

The same result can be obtained from the exact solution, which is valid for any strength of the gravitational field, we obtained in this paper.
The horizon curve is obtained by the condition $g^{-1}_{rr}(r_H) = 0$

$$M = \frac{r_H}{2G_N}\frac{1}{1+\Gamma_U\,\,(\,R_*/r_H\,)^{2d_U-2}} \tag{26}$$

For $d_U = 1$ the horizon radius result to be increased with respect to the pure-gravity case as one finds

$$r_H = R_s\,(\,1+\Gamma_1\,) \tag{27}$$

The Hawking temperature is

$$T_{d_U} = \frac{1}{4\pi r_H}\frac{1}{\left[\,1+\Gamma_U\,(\,R_*/r_H\,)^{2d_U-2}\right]}\left[\,1+(\,2d_U-1\,)\,\Gamma_U\,\left(\frac{R_*}{r_H}\right)^{2d_U-2}\right]$$

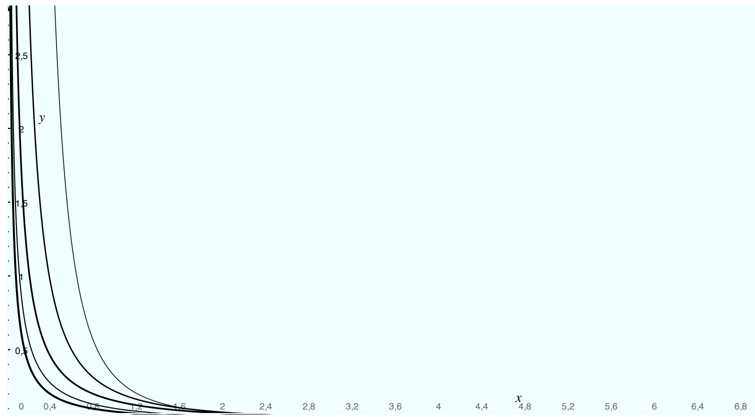

Fig. 2. *Plot of the function* $T_{d_U}(r_H)$ *for different values of* $d_U = 1\ , 1.25\ , 1.5\ , 1.75\ , 2.$

$$= \frac{1}{4\pi r_H}\left[1 + \frac{2\,(\,2d_u - 1\,)\;\Gamma_U}{1 + \Gamma_U\ (\,R_*/r_H\,)^{2d_U-2}}\ \left(\frac{R_*}{r_H}\right)^{2d_U-2}\right] \tag{28}$$

We can distinguish two different "phases" of the model :
i) *gravity-dominated* phase, where T_{d_U} takes the standard form

$$T_{d_U} \simeq T_H = \frac{1}{4\pi r_H} \tag{29}$$

ii) *un-gravity-dominated* phase, where T_{d_U} turns into

$$T_{d_U} \simeq \frac{2d_U - 1}{4\pi r_H} \tag{30}$$

It is interesting to compare Eq.(30) with the corresponding temperature for a Schwarzschild black hole in D spacetime dimensions

$$T_D \simeq \frac{D-3}{4\pi r_H} \tag{31}$$

More precisely, Eq.(31) gives the intrinsic temperature of a $D-2$ dimensional horizon. By comparing Eq.(30) with Eq.(31) we see that in the ungravity dominated phase the thermodynamic behavior of the horizon corresponds to a an *effective, non-integer dimension* $d_H = 2d_U$. Under this respect, ungravity leads to a *fractalisation* of the event horizon. Let us elaborate this picture by investigating the Area Law.
It is "customary" nowadays to assume that the black hole entropy is 1/4 of the event horizon area in Planck units, forgetting that this is a *consequence* of the First Law of black hole thermodynamics and not a "dogma" to be blindly accepted.

Against this background, we start from

$$dM = T_{d_U} dS \tag{32}$$

and derive S for a ungravity dominated black hole. In this regime we can express M in terms of r_H as

$$M \simeq \frac{R_*}{2\Gamma_U} \left(r_H / R_* \right)^{2d_U - 1} \tag{33}$$

The First Law takes the form

$$dS = \frac{dM}{T_{d_U}} = \frac{4\pi\, r_H}{2d_U - 1} \frac{\partial M}{\partial r_H} dr_H \tag{34}$$

By integrating (34) we find

$$S = \frac{\pi\, R_*^{2-2d_U}}{d_U\, \Gamma_U}\, r_H^{d_H} \tag{35}$$

Eq.(35) represents the Area Law for an Un-Schwarzschild black hole. It is immediate to check that for $d_U \to 1$ we recover the standard form

$$S \to \pi\, r_H^2 = \frac{1}{4} A_H \tag{36}$$

Thus, we feel confident to interpret Eq.(35) as the extension of the Area Law for a fracatlised horizon of dimension $d_H = 2d_U$.
The analogy between tensor un-particle dynamics and physics in presence of fractal extra-dimension has been noted in [42] in relation with multiplicity, temperature profile and decay rate of TeV micro black holes.
Our result trace back these effects not to a formal analogy with black holes in higher dimensional spacetime, but to the fractal geometry of the event horizon itself. The out-coming picture is that if we imagine the event horizon as a null surface "built-up" of un-gravitons trapped at the Schwarzschild radius, the underlying scale invariance of the theory manifests itself in the form of fractality, or self-similarity, of the horizon.
The presence of a source in the effective Einstein equations is instrumental to evaluate un-gravity corrections. In this paper we considered the simplest case of a point-like source leading to an "eternal" black hole type solution. An interesting problem is the study a collapsing body in order to see if and how un-gravity can change the dynamics of the collapse itself. Even in the simplest case of a collapsing shell of matter, it is non-trivial to extend the Israel matching formalism to the case of our effective, non-local, theory. Thus,

we postpone this study to a future investigation. A further prospect of pushing forward the study of un-particle fields in connection with gravity, we believe it would be interesting to build up an un-gravity action taking into account new gravitational degrees of freedom which drop out whenever we allow for dynamical torsion. This would lead to a richer spectrum and the un-particle sector might present some peculiar properties.

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